

# The Generalised Nikaido-Isoda Theorem

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## Preface

The main contribution of this typoscript is a variant of a theorem in [6] of Nikaido and Isoda. I refer to it as the “Generalized Nikaido-Isoda Theorem”. The variant is proven by dealing with, what i refer to as, “domination sets”. In addition to the Generalized Nikaido-Isoda Theorem the typoscript contains various related results wherein such sets play a role. These results may be useful when dealing with Nash equilibria of games in strategic form where strategy sets may be not compact. No result here should be considered as new. But i am not able to give appropriate references.

Of course, some prior knowledge is required: what that entails will quickly become clear when browsing through the typoscript.

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The author, of course, welcomes further comments and suggestions that may lead to improvements.

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<sup>1</sup>The (idealistic) public domain typesetting system LaTeX was used for its production under the(idealistic) operating system Linux.

# 1 Setting

Let us quickly recall basic game theoretic terminology. We deal with  $n$  player games in strategic form where  $N := \{1, \dots, n\}$  is the set of players and for all  $i \in N$ ,  $X_i$  is player  $i$ 's strategy set and  $f_i$  is player  $i$ 's payoff function. Henceforth,  $X_i$  is a non-empty set and

$$f_i : \mathbf{X} \rightarrow \mathbb{R}$$

is a function where  $\mathbf{X} := X_1 \times \dots \times X_n$ . We denote such a game also by  $(X_1, \dots, X_n; f_1, \dots, f_n)$ .

For  $i \in N$ , let  $\mathbf{X}_i := X_1 \times \dots \times X_{i-1} \times X_{i+1} \times \dots \times X_n$ . We sometimes identify  $\mathbf{X}$  with  $X_i \times \mathbf{X}_i$  and then write  $\mathbf{x} \in \mathbf{X}$  as  $\mathbf{x} = (x_i; \mathbf{x}_i)$ .

For a player  $i$  and  $\mathbf{z} \in \mathbf{X}_i$ , i.e. strategy profile  $\mathbf{z}$  of the other players, denote by  $f_i^{(\mathbf{z})}$ , the conditional payoff function of player  $i$  against  $\mathbf{z}$ , i.e. the function  $f_i^{(\mathbf{z})} : X_i \rightarrow \mathbb{R}$  given by

$$f_i^{(\mathbf{z})}(x_i) := f_i(x_i; \mathbf{z}).$$

For  $i \in N$ , the correspondence  $R_i : \mathbf{X}_i \multimap X_i$  is defined by

$$R_i(\mathbf{z}) := \operatorname{argmax} f_i^{(\mathbf{z})}.$$

The joint best-response correspondence is the correspondence  $\mathbf{R} : \mathbf{X} \multimap \mathbf{X}$  defined by

$$\mathbf{R}(\mathbf{x}) := R_1(\mathbf{x}_{\hat{1}}) \times \dots \times R_n(\mathbf{x}_{\hat{n}})$$

So, for  $i \in N$ ,  $R_i(\mathbf{X}_i)$  is the set of best-responses of player  $i$ . When  $X_i$  is a metric space, the correspondence  $R_i$  is called bounded if the set  $R_i(\mathbf{X}_i)$  is a bounded subset of  $X_i$ .

Denote by  $E$  the Nash equilibrium set, i.e. the set of  $\mathbf{x} \in \mathbf{X}$  with the property that for all  $i \in N$ ,  $x_i$  is a maximiser of  $f_i^{(\mathbf{x}_i)}$ . Also:  $E$  is the set of fixed points of  $\mathbf{R}$ . If the game is  $\Gamma$ , then we also denote this set by  $E(\Gamma)$ .

A strategy  $x_i \in X_i$  is said to be strongly dominated by a strategy  $d_i \in X_i$  if  $f_i^{(\mathbf{z})}(d_i) > f_i^{(\mathbf{z})}(x_i)$  for each  $\mathbf{z} \in \mathbf{X}_i$ .

Finally, given a correspondence  $F : A \multimap B$ , I say that  $F$  is proper if  $\#F(a) \geq 1$  for all  $a \in A$ . And if  $F$  is single-valued, i.e.  $\#F(a) = 1$  for all  $a \in A$ , then we also interpret  $F$  as a mapping  $F : A \rightarrow B$  by identifying for every  $a \in A$  the set  $F(a)$  with its unique element.

# 2 Maximisers of a restricted function

The next lemma presents useful simple results about the relation of the set of maximisers of the restriction of a function and the set of the maximisers of the unrestricted function.

**Lemma 1** *Suppose  $X$  is a set,  $g : X \rightarrow \mathbb{R}$  is a function and  $W$  is a non-empty subset of  $X$ . Then:*

1. (a)  $[a \in \operatorname{argmax} g \wedge a \in W] \Rightarrow a \in \operatorname{argmax} g|_W$ .

(b) *Sufficient for*

$$\operatorname{argmax} g \subseteq \operatorname{argmax} g|_W$$

*to hold is that  $\operatorname{argmax} g \subseteq W$ .*

(c) *Sufficient for*

$$\operatorname{argmax} g = \operatorname{argmax} g|_W$$

*to hold is that  $\emptyset \neq \operatorname{argmax} g \subseteq W$ .<sup>2</sup>*

2. *Suppose  $\sup g|_{(W \setminus X)} \leq \sup g|_W$ , i.e.  $g(x) \leq \sup g|_W$  ( $x \in X \setminus W$ ). Then:*

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<sup>2</sup>The assumption  $\emptyset \neq \operatorname{argmax} g$  is important:  $g(x) = x$  ( $x \in \mathbb{R}$ ),  $W = [0, 137]$ .

- (a)  $\operatorname{argmax} g|_W \subseteq \operatorname{argmax} g$ .
- (b) If  $\operatorname{argmax} g|_W \neq \emptyset$ , then each of the following three conditions separately is sufficient for  $\operatorname{argmax} g = \operatorname{argmax} g|_W$  to hold:
- i.  $\#\operatorname{argmax} g = 1$ .
  - ii.  $g(x) < \sup g|_W$  ( $x \in X \setminus W$ ).
  - iii.  $\sup g|(W \setminus X) < \sup g|_W$ .  $\diamond$

**Proof.**— 1a. Suppose  $a \in \operatorname{argmax} g$  and  $a \in W$ . So  $g(x) \leq g(a)$  ( $x \in X$ ). Thus also  $g(x) \leq g(a)$  ( $x \in W$ ). As  $a \in W$ ,  $a \in \operatorname{argmax} g|_W$  follows.

1b. By part 1a.

1c. By part 1b, it is sufficient to prove “ $\supseteq$ ”. So suppose  $b \in \operatorname{argmax} g|_W$ . So  $b \in W$  and  $g(x) \leq g(b)$  ( $x \in W$ ). By assumption,  $\operatorname{argmax} g \neq \emptyset$ . Fix  $a \in \operatorname{argmax} g$ . As, also by assumption,  $a \in W$ , we have  $g(a) \leq g(b)$ . As  $a \in \operatorname{argmax} g$ , we have  $g(b) \leq g(a)$ . Thus  $g(b) = g(a)$  and therefore also  $b \in \operatorname{argmax} g$ .

2a. Suppose  $a \in \operatorname{argmax} g|_W$ , so  $a$  is a maximiser of  $g|_W$  and therefore  $\max g|_W = \sup g|_W = g(a)$ . As for every  $x \in X \setminus W$ , also  $g(x) \leq \sup g|(X \setminus W) \leq \sup g|_W = g(a)$ ,  $a$  even is a maximiser of  $g$ .

2bi. The proof is complete by part 1a if we show that  $\operatorname{argmax} g \subseteq W$ . This we do by contradiction. So suppose  $\operatorname{argmax} g \not\subseteq W$ . As  $\#\operatorname{argmax} g = 1$ ,  $\operatorname{argmax} g$  is a singleton, say  $x$ . Fix  $a \in \operatorname{argmax} g|_W$ . Now  $\sup g|_W = g(a)$ . As  $\{x\} \not\subseteq W$ , we have  $x \in X \setminus W$  and therefore,  $g(x) \leq \sup g|(X \setminus W) \leq \sup g|_W = g(a)$ . Of course, also  $g(x) \geq g(a)$ . So  $g(x) = g(a)$ . As  $x$  is the unique maximiser of  $g$ , we have  $x = a \in W$ , a contradiction.

ii. The proof is complete by part 1a if we show that  $\operatorname{argmax} g \subseteq W$ . This we do by contradiction. So suppose  $\operatorname{argmax} g \not\subseteq W$ . Fix  $x \in \operatorname{argmax} g$  with  $x \notin W$ . Now  $g(x) < \sup g|_W$ , which is absurd.

iii. By part 2bii. □

### 3 Best-response supersets and domination sets

**Definition 1** Consider a game in strategic form. Let  $i \in N$ . A subset  $K_i$  of  $X_i$  is a best-response superset for player  $i$  if  $R_i(\mathbf{X}_i) \subseteq K_i$ .  $\diamond$

Note that  $X_i$  and  $R_i(\mathbf{X}_i)$  are best-response supersets for player  $i$  and that the empty set also may be a best-response superset. Also note that if  $K_i$  is a best-response superset for player  $i$  and  $K'_i \supseteq K_i$ , then also  $K'_i$  is a best-response superset for that player.

**Proposition 1** Consider a game in strategic form. Let  $K_i$  be a best-response superset for player  $i$  and  $\mathbf{z} \in \mathbf{X}_i$ . If  $R_i(\mathbf{z}) \neq \emptyset$ , then  $R_i(\mathbf{z}) = \operatorname{argmax} f_i^{(\mathbf{z})} = \operatorname{argmax} f_i^{(\mathbf{z})}|_{K_i}$ .  $\diamond$

**Proof.**— Suppose  $R_i(\mathbf{z}) \neq \emptyset$ , so  $\operatorname{argmax} f_i^{(\mathbf{z})} \neq \emptyset$ . Lemma 1(1c) implies the desired result as  $\emptyset \neq \operatorname{argmax} f_i^{(\mathbf{z})} \subseteq K_i$ . □

#### 3.1 Domination sets

**Definition 2** Consider a game in strategic form. Let  $i \in N$ . A subset  $K_i$  of  $X_i$  is a domination set for player  $i$  if  $K_i \neq \emptyset$  and every  $x_i \in X_i \setminus K_i$  is strongly dominated by some element of  $K_i$ .  $\diamond$

Note that  $X_i$  is domination set. Also note that if  $K_i$  is a domination set for player  $i$  and  $K'_i \supseteq K_i$ , then also  $K'_i$  is a domination set for that player.

**Lemma 2** Consider a game in strategic form. Let  $i \in N$ . Suppose  $K_i$  is a domination set for player  $i$ . Then for all  $\mathbf{z} \in \mathbf{X}_i$  it holds that  $f_i^{(\mathbf{z})}(x_i) < \sup f_i^{(\mathbf{z})}|_{K_i}$  ( $x_i \in X_i \setminus K_i$ ).  $\diamond$

**Proof.**— Suppose  $\mathbf{z} \in \mathbf{X}_i$  and  $x_i \in X_i \setminus K_i$ . By definition,  $x_i$  is strongly dominated by some element of  $K_i$ , say by  $d_i$ . This implies  $f_i^{(\mathbf{z})}(x_i) < f_i^{(\mathbf{z})}(d_i)$ . It follows that  $f_i^{(\mathbf{z})}(x_i) < f_i^{(\mathbf{z})}(d_i) \leq \sup f_i^{(\mathbf{z})} \upharpoonright K_i$ .  $\square$

**Proposition 2** *Consider a game in strategic form. Let  $i \in N$ . Suppose  $K_i$  is a domination set for player  $i$ . Then  $R_i(\mathbf{X}_i) \subseteq K_i$ , i.e.  $K_i$  is a best-response superset for player  $i$ .  $\diamond$*

**Proof.**— Fix  $\mathbf{z} \in \mathbf{X}_i$ . We show by contradiction that  $R_i(\mathbf{z}) \subseteq K_i$ . So suppose  $a_i \in R_i(\mathbf{z}) = \operatorname{argmax} f_i^{(\mathbf{z})}$  with  $a_i \notin K_i$ . As  $K_i$  is a domination set for player  $i$ ,  $a_i$  is strongly dominated by some  $d_i \in K_i$ . So  $f_i^{(\mathbf{z}')} (d_i) > f_i^{(\mathbf{z}')} (a_i)$  for all  $\mathbf{z}' \in \mathbf{X}_i$ . In particular  $f_i^{(\mathbf{z})} (d_i) > f_i^{(\mathbf{z})} (a_i)$ , which is a contradiction with  $a_i$  being a maximiser of  $f_i^{(\mathbf{z})}$ .  $\square$

**Proposition 3** *Consider a game in strategic form. Let  $i \in N$ ,  $K_i$  a domination set for  $i$  and  $\mathbf{z} \in \mathbf{X}_i$ . Then  $R_i(\mathbf{z}) = \operatorname{argmax} f_i^{(\mathbf{z})} = \operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright K_i$ .  $\diamond$*

**Proof.**— By Proposition 2,  $K_i$  is a best-response superset for player  $i$ . So if  $R_i(\mathbf{z}) \neq \emptyset$ , then the desired result follows from Proposition 1.

Next suppose  $R_i(\mathbf{z}) = \emptyset$ . By Lemma 2, it holds that  $f_i^{(\mathbf{z})}(x_i) < \sup f_i^{(\mathbf{z})} \upharpoonright K_i$  ( $x_i \in X_i \setminus K_i$ ). So Lemma 1(2a), implies, as desired,  $\operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright K_i \subseteq \operatorname{argmax} f_i^{(\mathbf{z})} = \emptyset$ .  $\square$

Note that in case  $X_i$  is a metric space player  $i$  has a bounded best-response superset if and only if the best-response correspondence  $R_i$  is bounded.

**Lemma 3** *Consider a game in strategic form. Let  $i \in N$  and suppose the strategy set  $X_i$  is a closed subset of a finite dimensional normed real linear space.*

1. *Suppose player  $i$  has a bounded best-response superset. Then*

(a) *Player  $i$  has a compact best-response superset.*

(b) *If  $X_i$  is convex, then player  $i$  has a convex compact best-response superset.*

2. *Suppose player  $i$  has a bounded domination set. Then*

(a) *Player  $i$  has a compact domination set.*

(b) *If  $X_i$  is convex, then player  $i$  has a convex compact domination set.  $\diamond$*

**Bewijs.**— Let  $E$  be the linear space.

1a. Let  $W_i$  be a bounded best-response superset.. As  $W_i$  is a bounded subset of  $X_i$ ,  $W_i$  is a bounded subset of  $E$ . Let  $K_i$  be its topological closure. Also  $K_i$  is bounded. As  $E$  is finite dimensional, it follows that  $K_i$  is compact. As  $W_i \subseteq X_i$  and  $X_i$  is closed, also  $K_i$  is a subset of  $X_i$ . As  $K_i$  contains  $W_i$ , it follows that  $K_i$  is a compact best-response superset for player  $i$ .

1b. Suppose  $X_i$  is convex. By part 1a, let  $K_i$  be a compact best-response superset. As  $E$  is finite dimensional, this implies that also  $\operatorname{Conv}(K_i)$  is compact. As  $K_i \subseteq X_i$  and  $X_i$  is convex, also  $\operatorname{Conv}(K_i)$  is a subset of  $X_i$ . So  $\operatorname{Conv}(K_i)$  is a convex compact best-response superset.

2a. Now player  $i$  has a bounded best-response superset, say  $L_i$ . As  $L_i$  is a bounded subset of  $X_i$ ,  $L_i$  is a bounded subset of  $E$ . Let  $K_i$  be its topological closure. Also  $K_i$  is bounded. As  $E$  is finite dimensional, it follows that  $K_i$  is compact. As  $L_i \subseteq X_i$  and  $X_i$  is closed, also  $L_i$  is a subset of  $X_i$ . As  $K_i$  contains  $L_i$ , it follows that  $K_i$  is a compact domination set for player  $i$ .

2b. Suppose  $X_i$  is convex. By part 2a, let  $K_i$  be a compact domination set. As  $E$  is finite dimensional, this implies that also  $\operatorname{Conv}(K_i)$  is compact. As  $K_i \subseteq X_i$  and  $X_i$  is convex, also  $\operatorname{Conv}(K_i)$  is a subset of  $X_i$ . So  $\operatorname{Conv}(K_i)$  is a convex compact best-response superset.  $\square$

**Lemma 4** *Consider a game in strategic form. Let  $i \in N$  and suppose the strategy set  $X_i$  is a subset of a normed real linear space. Suppose that for player  $i$  the conditional payoff functions are continuous and that player  $i$  has a compact domination set. Then*

1. The best-response correspondence  $R_i$  is proper.
2. Suppose  $X_i$  is convex and the conditional payoff functions of player  $i$  are strictly quasi-concave. Then  $R_i$  is single-valued.  $\diamond$

**Proof.**— 1. Let  $K_i$  be a compact domination set. By Proposition 3,  $R_i(\mathbf{z}) = \operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright_{K_i}$  ( $\mathbf{z} \in \mathbf{X}_i$ ). Fix  $\mathbf{z} \in \mathbf{X}_i$ . As also  $f_i^{(\mathbf{z})} \upharpoonright_{K_i}$  is continuous and  $K_i$  is non-empty and compact,  $\#R_i(\mathbf{z}) \geq 1$  follows.

2. Fix  $\mathbf{z} \in \mathbf{X}_i$ . We show that  $\#R_i(\mathbf{z}) = 1$ . By part 1,  $\#R_i(\mathbf{z}) \geq 1$ . As  $f_i^{(\mathbf{z})}$  is strictly quasi-concave,  $\#R_i(\mathbf{z}) \leq 1$  follows.  $\square$

## 4 Restricted games

**Lemma 5** Consider a game in strategic form

$$\Gamma = (X_1, \dots, X_n; f_1, \dots, f_n).$$

Suppose for every player  $i$  that  $K_i$  is a non-empty subset of  $X_i$ . Let  $\Gamma'$  be the game in strategic form

$$\Gamma' = (K_1, \dots, K_n; g_1, \dots, g_n)$$

where  $g_i := f_i \upharpoonright_{\mathbf{K}}$ . Then  $E(\Gamma) \cap \mathbf{K} \subseteq E(\Gamma')$ .  $\diamond$

**Proof.**— Suppose  $\mathbf{e} \in E(\Gamma) \cap \mathbf{K}$ , i.e. for every  $i$  we have  $e_i \in K_i$  and  $e_i$  is a maximiser of  $f_i^{(\mathbf{e}_i)}$ . Lemma 1(1) guarantees that  $e_i$  also is a maximiser of  $f_i^{(\mathbf{e}_i)} \upharpoonright_{K_i}$ . Thus  $\mathbf{e} \in E(\Gamma')$ .  $\square$

The result in Lemma 5 becomes more nice if every  $K_i$  is a non-empty best-response superset:

**Proposition 4** Consider a game in strategic form

$$\Gamma = (X_1, \dots, X_n; f_1, \dots, f_n).$$

Suppose for every player  $i$  that  $K_i$  is a non-empty best-response superset. Let  $\Gamma'$  be the game in strategic form

$$\Gamma' = (K_1, \dots, K_n; g_1, \dots, g_n)$$

where  $g_i := f_i \upharpoonright_{\mathbf{K}}$ . For  $i \in N$ , let  $R'_i$  be the best-response correspondence of player  $i$  in the game  $\Gamma'$ . Then:

1. For every  $i \in N$  and  $\mathbf{z} \in \mathbf{K}_i$ :  $R_i(\mathbf{z}) \neq \emptyset \Rightarrow R_i(\mathbf{z}) = R'_i(\mathbf{z})$ .
2.  $E(\Gamma) \subseteq E(\Gamma')$ .
3. If each best-response correspondence  $R_i$  is proper, then  $E(\Gamma) = E(\Gamma')$ .  $\diamond$

**Proof.**— First note that, for  $\mathbf{z} \in \mathbf{K}_i$ ,  $R'_i(\mathbf{z}) = \operatorname{argmax} g_i^{(\mathbf{z})} = \operatorname{argmax} (f_i \upharpoonright_{\mathbf{K}})^{(\mathbf{z})} = \operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright_{K_i}$ .

1. Suppose  $R_i(\mathbf{z}) \neq \emptyset$ . By Proposition 1,  $R_i(\mathbf{z}) = \operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright_{K_i} = R'_i(\mathbf{z})$ .

2. Suppose  $\mathbf{e} \in E(\Gamma)$ . We prove that  $\mathbf{e} \in \mathbf{K}$ , i.e. that  $e_i \in K_i$  ( $i \in N$ ) and then the proof is complete by Lemma 5. Well, fix  $i$ . Note that  $e_i \in R_i(\mathbf{e}_i)$ . As  $K_i$  is a best-response superset,  $e_i \in K_i$  holds.

3. Suppose each correspondence  $R_i$  is proper. By part 2, we still have to prove “ $\supseteq$ ”. So suppose  $\mathbf{e} \in E(\Gamma')$ . I.e., for every  $i \in N$ ,  $\mathbf{e}_i \in \mathbf{K}_i$  and  $e_i \in R'_i(\mathbf{e}_i)$ . As every  $R_i$  is proper,  $R_i(\mathbf{e}_i) \neq \emptyset$  ( $i \in N$ ) holds. Now part 1 implies, as desired,  $e_i \in R'_i(\mathbf{e}_i) = R_i(\mathbf{e}_i)$  ( $i \in N$ ).  $\square$

## 5 Continuity of best-response correspondences

**Proposition 5** *Consider a game in strategic form where each strategy set is a metric space. Let  $i$  be a player. Suppose the strategy set  $X_i$  is compact, the payoff function  $f_i$  is continuous and the best-response correspondence  $R_i : \mathbf{X}_{-i} \multimap X_i$  is single-valued. Then the mapping  $R_i : \mathbf{X}_{-i} \rightarrow X_i$  is continuous.  $\diamond$*

**Proof.**— A comfortable way to prove this, is with Berge’s maximum theorem, i.e. with Theorem 5 in the appendix. Let us check its conditions (we apply it with “ $F(\mathbf{z}) = X_i$ ”).

Well,  $X_i$  and  $\mathbf{X}_{-i}$  are non-empty metric spaces,  $f_i : X_i \times \mathbf{X}_{-i} \rightarrow \mathbb{R}$  is a continuous function and  $X_i$  is compact. For  $\mathbf{z} \in \mathbf{X}_{-i}$ , letting

$$m(\mathbf{z}) := \max_{x_i \in X_i} f_i^{(\mathbf{z})}(x_i),$$

we have  $R_i(\mathbf{z}) = \{x_i \in X_i \mid f_i^{(\mathbf{z})}(x_i) = m(\mathbf{z})\}$ . Theorem 5(3) guarantees that  $R_i$  is continuous.  $\square$

Here is a proof “by hand” of this proposition: fix  $\mathbf{z} \in \mathbf{X}_{-i}$ . We prove that  $R_i$  is continuous at  $\mathbf{z}$ . In order to do so we take a sequence  $(\mathbf{z}_m)$  in  $\mathbf{X}_{-i}$  with  $\lim_{m \rightarrow \infty} \mathbf{z}_m = \mathbf{z}$  and show that, with  $b_m := R_i(\mathbf{z}_m) \in X_i$ , the sequence  $(b_m)$  is convergent with  $\lim_{m \rightarrow \infty} b_m = R_i(\mathbf{z})$ .

In order to do so we first prove that each convergent subsequence of  $(b_m)$  has limit  $R_i(\mathbf{z})$ . Well let  $(b_{p(m)})$  be a convergent subsequence, say its limit is  $b_\star \in X_i$ . For any  $m$ , we have  $f_i(b_{p(m)}; \mathbf{z}_{p(m)}) \geq f_i(x_i; \mathbf{z}_{p(m)})$  ( $x_i \in X_i$ ). These inequalities together with the continuity of  $f_i$  imply  $f_i(b_\star; \mathbf{z}) \geq f_i(x_i; \mathbf{z})$  ( $x_i \in X_i$ ). Thus  $b_\star \in R_i(\mathbf{z})$ . As  $\#R_i(\mathbf{z}) = 1$ , it follows, as desired, that  $b_\star = R_i(\mathbf{z})$ .

Further, as  $X_i$  is compact, the sequence  $(b_m)$  has a convergent subsequence  $(b_{p(m)})$ . So we have shown that each convergent subsequence of  $(b_m)$  has the same limit, i.e.  $R_i(\mathbf{z})$ . In other words: the sequence of  $(b_m)$  has  $R_i(\mathbf{z})$  as unique limit point. This implies that the sequence  $(b_m)$  is convergent with  $\lim_{m \rightarrow \infty} b_m = R_i(\mathbf{z})$ .

Remembering that  $X_i$  is a best-response superset for player  $i$ , we see that the following proposition improves upon the previous one.

**Proposition 6** *Consider a game in strategic form where each strategy set is a metric space. Let  $i$  be a player. Suppose the payoff function  $f_i$  is continuous and the best-response correspondence  $R_i : \mathbf{X}_{-i} \multimap X_i$  is single-valued. Then each of the following two conditions separately is sufficient for the mapping  $R_i : \mathbf{X}_{-i} \rightarrow X_i$  to be continuous.*

- a. *There exists a compact best-response superset for player  $i$ .*
- b. *The strategy set  $X_i$  is a closed subset of a finite dimensional real normed linear space  $E$  and the best-response mapping  $R_i : \mathbf{X}_{-i} \rightarrow X_i$  is bounded.  $\diamond$*

**Proof.**— a. Let  $K_i$  be a compact best-response superset for player  $i$ . As the correspondence  $R_i$  is single-valued,  $K_i$  is not empty. As for every  $\mathbf{z} \in \mathbf{X}_{-i}$ , we have  $R_i(\mathbf{z}) \neq \emptyset$ , Proposition 1 implies for every  $\mathbf{z} \in \mathbf{X}_{-i}$  that

$$R_i(\mathbf{z}) = \operatorname{argmax} f_i^{(\mathbf{z})} = \operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright K_i.$$

This makes that again we can apply Berge’s maximum theorem for obtaining the desired result: now apply it to the restriction of  $f_i$  to  $K_i \times \mathbf{X}_{-i}$ . In doing so it follows that mapping  $R_i : \mathbf{X}_{-i} \rightarrow K_i$  is continuous. So also the mapping  $R_i : \mathbf{X}_{-i} \rightarrow X_i$  is continuous.

- b. By Lemma 3(1), there exists a compact best-response superset. Now apply part a.  $\square$

**Remark:** the two previous propositions were proved by referring to Berge’s maximum theorem. Although this theorem supposes continuity of the function that is, for each parameter value, maximized, it does not suppose that for each parameter value there exists precisely one maximiser as we supposed in the two previous propositions.

The next proposition in turn improves upon the two above ones. It relies on a result that use the closedness of a graph.

**Theorem 1** Consider a game in strategic form where every action set is a metric space. Fix  $i \in N$ . Suppose  $f_i : \mathbf{X} \rightarrow \mathbb{R}$  is upper-semicontinuous and for every  $x_i \in X_i$  the function  $f_i^{(\cdot)}(x_i) : \mathbf{X}_i \rightarrow \mathbb{R}$  is lower-semicontinuous and the best-response correspondence  $R_i : \mathbf{X}_i \rightarrow X_i$  is single-valued. Then each of the following two conditions separately is sufficient for the mapping  $R_i : \mathbf{X}_i \rightarrow X_i$  to be continuous.

- a. There exists a compact best-response superset for player  $i$ .
- b. The strategy set  $X_i$  is a closed subset of a finite dimensional real linear space  $E$  and the best-response mapping  $R_i : \mathbf{X}_i \rightarrow X_i$  is bounded.  $\diamond$

**Proof.**— a. Let  $K_i$  be a compact best-response superset for player  $i$ . As the correspondence  $R_i$  is single-valued,  $K_i$  is not empty. If we show that the mapping  $R_i : \mathbf{X}_i \rightarrow K_i$  is continuous, then also  $R_i : \mathbf{X}_i \rightarrow X_i$  is continuous. Further consider  $R_i : \mathbf{X}_i \rightarrow K_i$ .

As  $K_i$  is a compact metric space and  $\mathbf{X}_i$  is a metric space, sufficient (and necessary) for  $R_i$  to be continuous is that  $R_i$  has a closed graph, i.e. that the set  $\{(\mathbf{z}, R_i(\mathbf{z})) \mid \mathbf{z} \in \mathbf{X}_i\}$  is closed in  $\mathbf{X}_i \times K_i$ . In order to show that this graph is closed, we take a convergent sequence  $(\mathbf{z}^{(m)}, R_i(\mathbf{z}^{(m)}))$  from this graph with

$$\lim_{m \rightarrow \infty} (\mathbf{z}^{(m)}, R_i(\mathbf{z}^{(m)})) = (\mathbf{z}, k_i) \in \mathbf{X}_i \times K_i$$

and show that  $k_i = R_i(\mathbf{z})$ , i.e. that  $f_i(k_i; \mathbf{z}) \geq f_i(y; \mathbf{z})$  ( $y \in K_i$ ). So fix  $y \in K_i$ . Note that  $\lim_{m \rightarrow \infty} \mathbf{z}^{(m)} = \mathbf{z}$ ,  $\lim_{m \rightarrow \infty} R_i(\mathbf{z}^{(m)}) = k_i$  and  $f_i(R_i(\mathbf{z}^{(m)}); \mathbf{z}^{(m)}) \geq f_i(y; \mathbf{z}^{(m)})$ . As  $f_i$  is upper-semicontinuous in  $(k_i; \mathbf{z})$  and  $f_i^{(\cdot)}(y)$  is lower-semicontinuous in  $\mathbf{z}$ , we obtain, as desired,

$$\begin{aligned} f_i(k_i; \mathbf{z}) &\geq \limsup_{m \rightarrow \infty} f_i(R_i(\mathbf{z}^{(m)}); \mathbf{z}^{(m)}) \geq \\ &\limsup_{m \rightarrow \infty} f_i(y; \mathbf{z}^{(m)}) \geq \liminf_{m \rightarrow \infty} f_i(y; \mathbf{z}^{(m)}) = \liminf_{m \rightarrow \infty} f_i^{(\mathbf{z}^{(m)})}(y) \\ &\geq f_i^{(\mathbf{z})}(y) = f_i(y; \mathbf{z}). \end{aligned}$$

- b. By Lemma 3(1), there exists a compact best-response superset. Now apply part a.  $\square$

## 6 Nash Equilibrium existence via Brouwer's fixed point theorem

**Proposition 7** Consider a game in strategic form where each strategy sets  $X_i$  is a closed convex subset of a finite dimensional normed real linear space and where best-response correspondences  $R_i$  are single-valued. If each best-response mapping  $R_i : \mathbf{X}_i \rightarrow X_i$  is continuous, then each of the following two conditions is sufficient for the game to have a Nash equilibrium.

- a. Each strategy set is compact.
- b. Each player has a bounded domination set.  $\diamond$

**Proof.**— Suppose each mapping  $R_i : \mathbf{X}_i \rightarrow X_i$  is continuous.

a. Also the joint best-response mapping  $\mathbf{R} : \mathbf{X} \rightarrow \mathbf{X}$  is continuous. As each  $X_i$  is a non-empty convex compact subset of, say,  $E_i$ , it follows that  $\mathbf{X}$  is a non-empty convex compact subset of  $E := E_1 \times \dots \times E_n$ . As  $E$  is finite dimensional, Brouwer's fixed point theorem guarantees that  $\mathbf{R}$  has a fixed point. Thus the game has a Nash equilibrium.

2. By Lemma 3(2b), each player  $i$  has a convex compact domination set  $K_i$ . Note that, by Proposition 2,  $K_i$  also is a best-response superset for player  $i$ . Using the notations  $\Gamma$  and  $\Gamma'$  of Proposition 4, the game  $\Gamma'$  is well-defined.

With the  $R'_i$  the best-response correspondences in  $\Gamma'$ , Proposition 4(1) implies  $R_i(\mathbf{z}) = R'_i(\mathbf{z})$  ( $\mathbf{z} \in \mathbf{K}_i$ ). Thus the  $R'_i$  are single-valued. As the  $R_i$  are continuous, the  $R'_i$  are continuous too.

Part a applies to  $\Gamma'$  and guarantees that  $E(\Gamma') \neq \emptyset$ . Now Proposition 4(3) guarantees that  $E(\Gamma) = E(\Gamma')$ . Thus  $E(\Gamma) = E(\Gamma') \neq \emptyset$ .  $\square$

**Theorem 2** *Consider a game in strategic form where each strategy set is a convex closed subset of a finite dimensional normed real linear space. Suppose payoff functions are continuous and conditional payoff functions are strictly quasi-concave. Then each of the following two conditions is sufficient for the game to have a Nash equilibrium.*

- a. *Each strategy set is compact.*
- b. *Each player has a bounded domination set.*  $\diamond$

**Proof.**— It is sufficient to prove part b. By Lemma 4(2), each best-response correspondence  $R_i$  is single-valued and thus a mapping  $R_i : \mathbf{X}_i \rightarrow X_i$ . By Proposition 2, each such mapping is bounded. Now Proposition 6(b) guarantees that these mappings are continuous. Proposition 7(b) guarantees that the game has a Nash equilibrium.  $\square$

## 7 Generalised Nikaido-Isoda Theorem

The following powerful Nash equilibrium existence result (cfr. with Theorem 2) being a variant of a result in [6] by Nikaido and Isoda, holds:

**Theorem 3** *A game in strategic form where the strategy sets are convex compact subsets of a finite dimensional normed real linear space, payoff functions are continuous and conditional payoff functions are quasiconcave has a Nash equilibrium.*  $\diamond$

**Proof.**— See [4] (en [3]). The proof there refers to Kakutani's fixed point theorem.  $\square$

Note that (because of the Weierstrass' theorem) best-response correspondences in this theorem are proper, but not necessarily single-valued.

Remark: the reason that the author refers to Theorem 3 as a variant of a result in [6] by Nikaido and Isoda, is that various other authors do this; see for example [4, 3, 5]. However, the article [2] already essentially contains this result.

Below we shall with Theorem 4 improve upon Theorem 3 concerning the compactness of strategy sets.

**Theorem 4** (*Generalized Nikaido-Isoda theorem.*) *Consider a game in strategic form with strategy sets that are closed convex subsets of a finite dimensional normed real linear space, continuous payoff functions and quasi-concave conditional payoff functions. Then each of the following two conditions separately is sufficient for the game to have a Nash equilibrium.*

- a. *Each player has a bounded domination set.*
- b. *Each strategy set is compact.*  $\diamond$

**Proof.**— a. By Lemma 3(2b) each player has a convex compact domination set, say  $K_i$ . Note that, by Proposition 2,  $K_i$  also is a best-response superset for player  $i$ . Using the notations  $\Gamma$  and  $\Gamma'$  of Proposition 4, the game  $\Gamma'$  is well-defined. Theorem 3 applies to  $\Gamma'$  and guarantees that  $E(\Gamma') \neq \emptyset$ . By Lemma 4(1), each best-response correspondence  $R_i$  is proper. Proposition 4(3) guarantees that  $E(\Gamma) = E(\Gamma')$ . Thus  $E(\Gamma) = E(\Gamma') \neq \emptyset$ .

- b. By part a, as a compact strategy set is a bounded domination set.  $\square$

## A Berge's maximum theorem

The following theorem concerns Berge's maximum theorem; e.g. see [1].

**Theorem 5** Let  $X$  and  $T$  be non-empty metric spaces,  $F : T \multimap X$  a compact-valued proper correspondence and  $f : X \times T \rightarrow \mathbb{R}$  a continuous function. Let the function  $m : T \rightarrow \mathbb{R}$  be well-defined by<sup>3</sup>

$$m(t) := \max_{x \in F(t)} f(x, t)$$

and let the compact-valued proper correspondence  $M : T \multimap X$  be well-defined by

$$M(t) := \{x \in F(t) \mid f(x, t) = m(t)\}.$$

1. Let  $t \in T$ . If  $F$  is continuous at  $t$ , then  $M$  is upper hemicontinuous at  $t$  and  $m$  is continuous at  $t$ .
2. If  $F$  is continuous, then  $M$  is hemicontinuous and  $m$  is continuous.
3. If  $F$  is continuous and  $M$  is single-valued, then  $M$  interpreted as a mapping  $T \rightarrow X$  is continuous.  $\diamond$

## B Brouwer's fixed point theorem

The following theorem is well-known. It is a deep result. Various proofs exist.

**Theorem 6** (*Brouwer's fixed point theorem*) Let  $D$  be a non-empty convex compact convex set in a finite dimensional normed real linear space and  $F : D \rightarrow D$  a continuous mapping. Then  $F$  has a fixed point.  $\diamond$

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<sup>3</sup>The well-definedness of  $m$  follows from the Weierstrass theorem and that of  $M$  concerns a general topological result about compactness of the set of maximisers of a continuous function with domain a compact metric space.